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NON-NEWTONIAN FLOW OVER A WEDGE WITH SUCTION

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SUMMARY

A pseudo-similarity solution is obtained for the flow of an incompressible fluid of second grade past a wedge with suction at the surface. The non-linear differential equation is solved using quasi-linearization and orthonormalization. The numerical method developed for this purpose enables computation of the flow characteristics for any values of the parameters K, a and b, where K is the dimensionless normal stress modulus of the fluid, a is related to the wedge angle and b is the suction parameter. A significant effect of suction on the wall shear stress is observed. The present results match exactly those from an earlier perturbation analysis for $Kx^{2a} \leqslant 0{\cdot}01$ but differ significantly as Kx^{2a} increases.

KEY WORDS Non-Newtonian Flow Wedge flow Suction

1. INTRODUCTION

Boundary layer flows of non-Newtonian fluids have wide-ranging applications, for example in the design of thrust bearings and radial diffusers, drag reduction, transpiration cooling and thermal oil recovery.[1] In the case of fluids of the differential type,[2] except for fluids of complexity $n=1$, the equations of motion are an order higher than the Navier–Stokes equations and the adherence boundary condition is insufficient to determine the solution completely. The same is also true for the appropriate boundary layer approximations of the equations of motion.

In the absence of a clear means of obtaining additional boundary conditions, Beard and Walters,[3] in their study of an incompressible fluid of second grade, suggested a perturbation approach in which the velocity and the pressure field were expanded in a series in terms of a small parameter ε; the parameter in question multiplied the highest-order spatial derivatives in their equation. While the assumption reduces the order of the equation, it treats a singular perturbation problem as though it were regular. In the case of flows which take place in unbounded domains, however, it is possible to augment the boundary conditions on the basis of the fact that the solution has to be bounded or has a certain smoothness at infinity.

Recently Garg and Rajagopal[4] studied the stagnation flow of a fluid of second grade by augmenting the boundary conditions. Their results agree well with the results of Rajeswari and Rathna[5] based on the perturbation approach for small values of the perturbation parameter. The advantage of augmenting the boundary conditions over the perturbation approach is that the analysis is valid even for large values of the parameter ε and, as shown by Garg and Rajagopal,[4] significant deviations from the Newtonian behaviour are possible for even moderately large values of ε.

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Herein we study the flow of a fluid of second grade past a wedge with suction by augmenting the boundary conditions. Recently Massoudi and Ramezan[1] established 'non-similar' solutions based on a perturbation approach for such a flow. We find that the results obtained by augmenting the boundary conditions are in perfect agreement with their results for small values of the perturbation parameter ε. We are able to obtain results for $\varepsilon \gg 1$ and in this case we see the striking effect of the non-Newtonian natue of the fluid and of the suction parameter on the skin friction.

2. BOUNDARY LAYER EQUATIONS

For a steady boundary layer flow of a fluid of second grade over a wedge (Figure 1), with u and v as the x- and y-components of the velocity field respectively, it follows from Reference 6 that

$$\frac{\partial \bar{u}}{\partial \bar{x}}+\frac{\partial \bar{v}}{\partial \bar{y}}=0, \tag{1}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}=\bar{U}\frac{\mathrm{d}\bar{U}}{\mathrm{d}\bar{x}}+\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}+K\left[\frac{\partial \bar{u}}{\partial \bar{y}}\frac{\partial^2 \bar{v}}{\partial \bar{y}^2}+\bar{v}\frac{\partial^3 \bar{u}}{\partial \bar{y}^3}+\frac{\partial}{\partial \bar{x}}\left(\bar{u}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right)\right]. \tag{2}$$

Here the dimensionless variables used are

$$\bar{u}=\frac{u}{U(0)}, \qquad \bar{v}=\frac{v}{U(0)}\sqrt{(Re)}, \qquad \bar{x}=\frac{x}{L}, \qquad \bar{y}=\frac{y}{L}\sqrt{(Re)},$$

$$\bar{U}=\frac{U}{U(0)}, \qquad Re=\frac{U(0)L}{\nu}, \qquad K=\frac{\alpha_1 Re}{\rho L^2},$$

with L as a characteristic length, ν as the kinematic viscosity and $U(x)$ as the potential flow velocity outside the boundary layer. With suction the boundary conditions are

$$\bar{u}=0 \quad \text{and} \quad \bar{v}=b\bar{x}^t \quad \text{at } \bar{y}=0, \qquad \bar{u}\to\bar{U} \quad \text{and} \quad \partial\bar{u}/\partial\bar{y}\to 0 \quad \text{as } \bar{y}\to\infty, \tag{3}$$

where a power-law mass injection velocity at the wedge surface has been assumed, with t as the injection velocity index and b as an appropriate dimensionless parameter. Clearly $b<0$ for suction at the surface.

Henceforth, for simplicity, we drop the overbar denoting the dimensionless quantities. By introducing a 'pseudo-similarity' variable

$$\eta=Cyx^a, \tag{4}$$

where C and a are constants, we seek a solution to the velocity field such that

$$u(x, y)=U(x)f'(\eta),$$

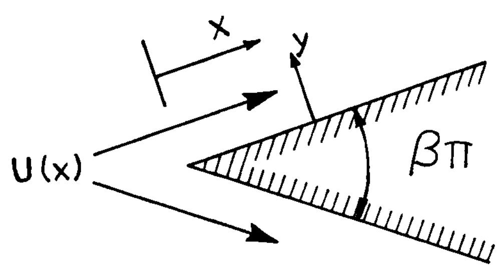

Figure 1. Wedge and co-ordinate system

and from (1)

$$v(x, y) = -\frac{U'}{Cx^{a}} f - \frac{aU}{Cx^{a+1}} (\eta f' - f),$$

where a prime denotes the derivative with respect to the argument. A straightforward substitution into (2) yields

$$K\left[\left(U' - \frac{aU}{x}\right) f f^{\mathrm{iv}} - \left(U' + \frac{aU}{x}\right)(2f'f''' - f''^{2})\right] - f'''$$
$$- f f''\left(\frac{U'}{C^{2}x^{2a}} - \frac{aU}{C^{2}x^{2a+1}}\right) - \frac{U'}{C^{2}x^{2a}}(1 - f'^{2}) = 0 \tag{5}$$

If $U = x^{2a+1}$ and $C^{2} = (a+1)$, equation (5) reduces to

$$Kx^{2a}[(a+1) f f^{\mathrm{iv}} - (3a+1)(2f'f''' - f''^{2})] - [f''' + f f'' + \beta(1 - f'^{2})] = 0, \tag{6}$$

where $\beta = (2a+1)/(a+1)$, $\beta\pi$ being the wedge angle. Notice that (6) involves both η and x and thus (2) has not been reduced to an ordinary differential equation as is usually the case with a similarity transformation. However, solving (6) is much easier than solving (2) since the former can be solved locally at a given x. When $a = 0$, equation (6) reduces to the equation governing the stagnation point flow studied by Garg and Rajagopal.[4] In this case equation (6) reduces to an ordinary differential equation, thus enabling a similarity solution. Also, for $K = 0$ equation (6) reduces to the well known Falkner–Skan equation for Newtonian flow over a wedge.

Since we are primarily interested in the case when $K \neq 0$, we shall rewrite (6) in the form

$$f^{\mathrm{iv}} = \frac{3a+1}{a+1} \frac{2f'f''' - f''^{2}}{f} + \frac{f''' + f f'' + \beta(1 - f'^{2})}{Kx^{2a}(a+1) f}. \tag{7}$$

The appropriate boundary conditions are

$$f(0) = -b(a+1)^{-1/2} x^{t-a}, \qquad f'(0) = 0, \qquad f'(\infty) = 1, \qquad f''(\infty) = 0. \tag{8}$$

Equations (7) and (8) are solved numerically using quasi-linearization and orthonormalization. We take $t = a$ for simplicity. The solution technique can, however, be used for $t \neq a$ as well.

3. NUMERICAL TECHNIQUE

The system of equations (7) and (8) constitutes a non-linear, non-homogeneous boundary value problem. It is solved locally at a given location using quasi-linearization and orthonormalization; the latter is required since the system is ill conditioned as well. The quasi-linearized form of (7) is[7]

$$(a+1) f_{n+1}^{\mathrm{iv}} - f_{n+1}'''[2f_n'(3a+1) + x^{-2a}/K]/f_n - f_{n+1}''[x^{-2a}/K - 2f_n''(3a+1)/f_n]$$
$$- f_{n+1}\{(3a+1)(f_n''^{2} - 2f_n'f_n''') - [\beta(1 - f_n'^{2}) + f_n''']x^{-2a}/K\}/f_n^{2}$$
$$- 2f_{n+1}'[f_n'''(3a+1) - \beta f_n' x^{-2a}/K]/f_n = (f_n''' + 2\beta)x^{-2a}/K f_n, \tag{9}$$

where the subscript n or $n+1$ represents the nth or $(n+1)$th approximation to the solution. Equation (9) being non-homogeneous, the solution at any level of approximation can be written as $f = f_{\mathrm{h}} + f_{\mathrm{p}}$, where f_{h} is the solution to the homogeneous part of (9) for the $(n+1)$th approximation and f_{p} is the particular solution. Since (9) is linear at every level of approximation, its homogeneous solution is a linear combination of two linearly independent solutions f_{h_1} and f_{h_2}, where $\{f_{\mathrm{h}_1}(0), f_{\mathrm{h}_1}'(0), f_{\mathrm{h}_1}''(0), f_{\mathrm{h}_1}'''(0)\} = \{0, 0, 1, 0\}$ and $\{f_{\mathrm{h}_2}(0), f_{\mathrm{h}_2}'(0), f_{\mathrm{h}_2}''(0), f_{\mathrm{h}_2}'''(0)\} = \{0, 0, 0, 1\}$.

The fourth order Runge–Kutta method is used to compute the two solutions f_{h_1} and f_{h_2}. Starting at $\eta=0$, the homogeneous part of (9) is integrated up to $\eta=\eta_\infty$, where the boundary conditions at $\eta=\infty$ are assumed to hold. The value $\eta_\infty=20$ was found to be adequate for all the suction cases ($b<0$) studied here. For the particular solution f_p the starting values are taken to be $f_p(0)=-b(a+1)^{-1/2}x^{t-a}$ and $f'_p(0)=f''_p(0)=f'''_p(0)=0$ and the Runge–Kutta method is used to integrate (9) up to η_∞.

Owing to the ill-conditioned nature of (9), the two solutions f_{h_1} and f_{h_2} that are orthogonal at $\eta=0$ become parallel as integration proceeds towards $\eta=\infty$. The two solutions therefore need to be orthonormalized.[8] The criterion used to decide when the two solution vectors need orthonormalization was based on the magnitude of the solution vectors. If the magnitude of any of the two vectors exceeded a preassigned constant M, orthonormalization was carried out. The constant M was taken to be 100. Needless to say, both the homogeneous solutions and the particular solution have to be corrected every time orthonormalization is carried out, as described in detail by Garg[8] and Scott and Watts.[9]

Once f_{h_1}, f_{h_2} and f_p were determined for $0\leqslant\eta\leqslant\eta_\infty$, the boundary conditions at $\eta=\infty$ were used to find the appropriate combination of f_{h_1} and f_{h_2}. In order to satisfy $f'(\infty)=1$ and $f''(\infty)=0$, we have

$$C_1 f'_{h_1}(\eta_\infty)+C_2 f'_{h_2}(\eta_\infty)=1-f'_p(\eta_\infty), \qquad C_1 f''_{h_1}(\eta_\infty)+C_2 f''_{h_2}(\eta_\infty)=-f''_p(\eta_\infty). \tag{10}$$

Knowing C_1 and C_2 from (10), we have the solution from

$$f(\eta)=C_1 f_{h_1}(\eta)+C_2 f_{h_2}(\eta)+f_p(\eta).$$

The zeroth approximation to the homogeneous solution was taken to be $f(\eta)=\eta(1-e^{-\eta})$. This satisfies all the boundary conditions for the homogeneous solution in (8), the non-zero value for $f(0)$ being recovered through the particular solution $f_p(0)$. Convergence was assumed when the ratio of any one of f, f', f'' or f''' for the last two approximations differed from unity by less than 10^{-5} at all values of η in $0<\eta<\eta_\infty$. Less than 10 approximations were required to satisfy this convergence criterion for all values of Kx^{2a} for which results were computed.

As many as 12 values of the step size $\Delta\eta$ were used, especially for large values of x or small values of Kx^{2a}, in order to reduce the number of points between $0\leqslant\eta\leqslant\eta_\infty$ without sacrificing accuracy. Actual values of $\Delta\eta$ ranged from 10^{-4} near $\eta=0$ to 0·5 near $\eta=\eta_\infty$. As a test of accuracy of the solution, it may be noted that for all solutions computed for $0{\cdot}01\leqslant Kx^{2a}\leqslant 200$, the value of $f^{\text{iv}}(0)$ computed from (7) at $\eta=0$ was compared with that obtained from the three-point forward difference relation

$$f^{\text{iv}}(0)=\frac{-3f'''_0+4f'''_1-f'''_2}{2\Delta\eta}+O[(\Delta\eta)^2], \tag{11}$$

where the subscripts 0, 1 and 2 refer to the values of f''' at $\eta=0$, $\Delta\eta$ and $2\Delta\eta$ respectively. The difference between the two values of $f^{\text{iv}}(0)$ thus computed was less than 10^{-6}. Also, $f'''(\eta_\infty)$ was found to be less than 10^{-10} in all cases. Moreover, results for $a=0$ (or $\beta=1$) and $b=0$ were exactly the same as those reported by Garg and Rajagopal.[4] We may also point out that for no suction ($b=0$) and $\beta=\frac{1}{2}$ or $a=-\frac{1}{3}$, $f'''(0)=-\frac{1}{2}$ independently of x and K. Our numerical solution verified this result exactly.

4. RESULTS AND DISCUSSION

Results were obtained for four values of β, namely $\beta=0$, 0·25, 0·5 and 1·0, various negative values of b and for Kx^{2a} varying from zero to 200. When $K=0$, the order of the differential equation (7)

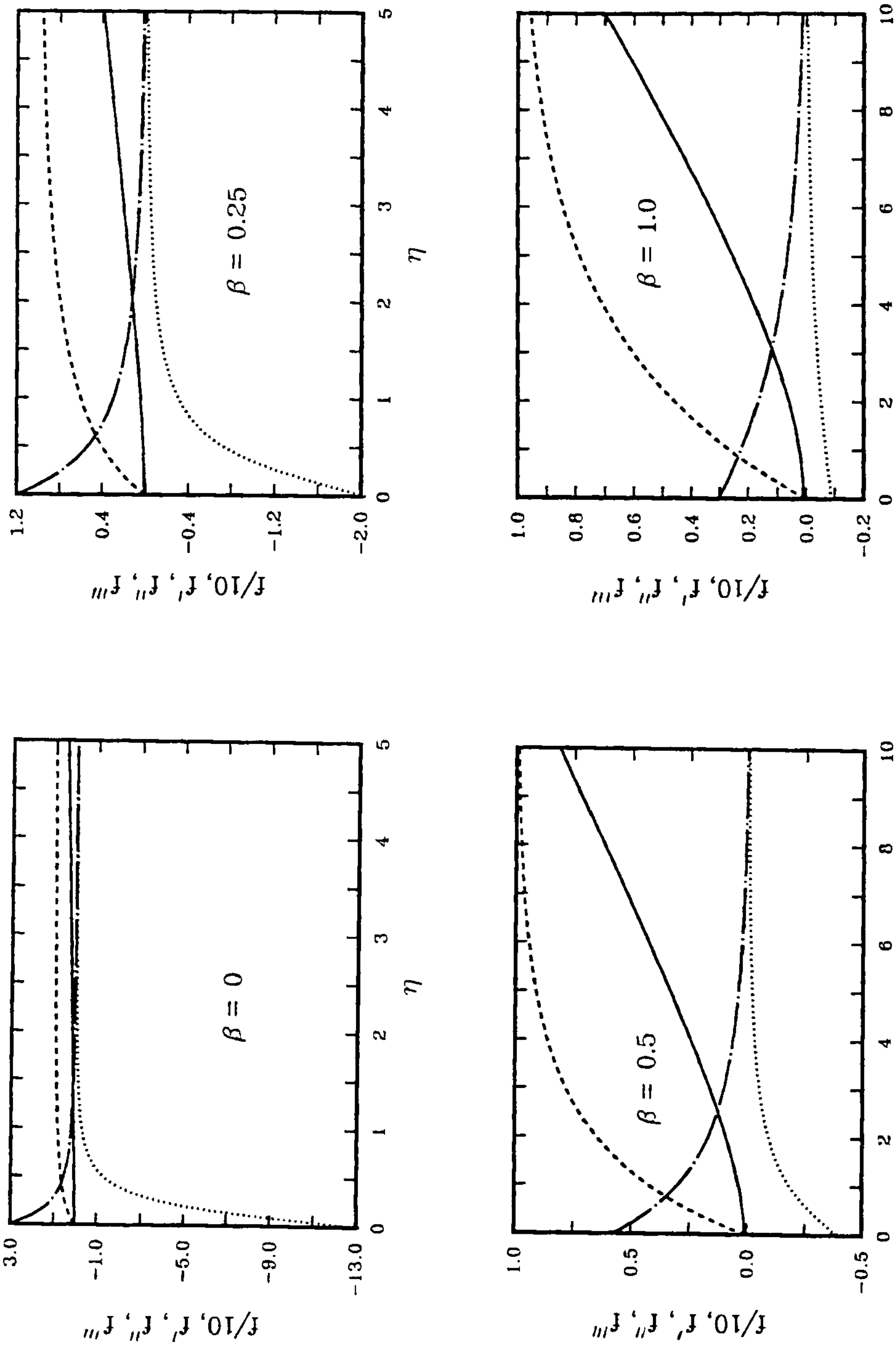

Figure 2. The function f and its derivatives for $x^{-2a}/K = 0{\cdot}1$, $b = -0{\cdot}1$ and $\beta = 0, \frac{1}{4}, \frac{1}{2}$ and 1: ——, $f/10$; - - -, f'; -·-·-, f''; . . . , f'''

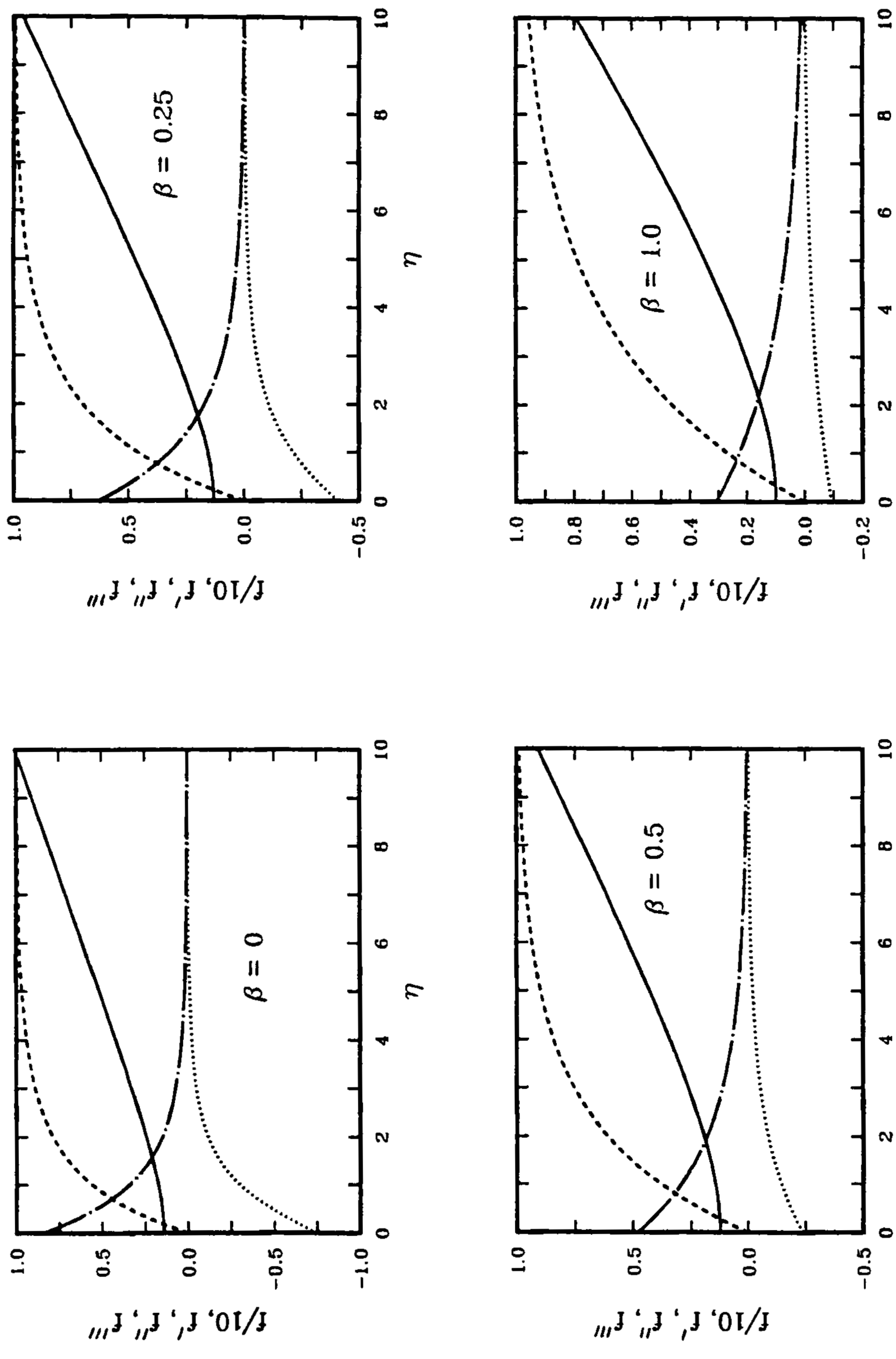

Figure 3. The function f and its derivatives for $x^{-2a}/K = 0{\cdot}1$, $b = -1$ and $\beta = 0, \frac{1}{4}, \frac{1}{2}$ and 1: ——, $f/10$; - - -, f'; - · -, f''; · · ·, f'''

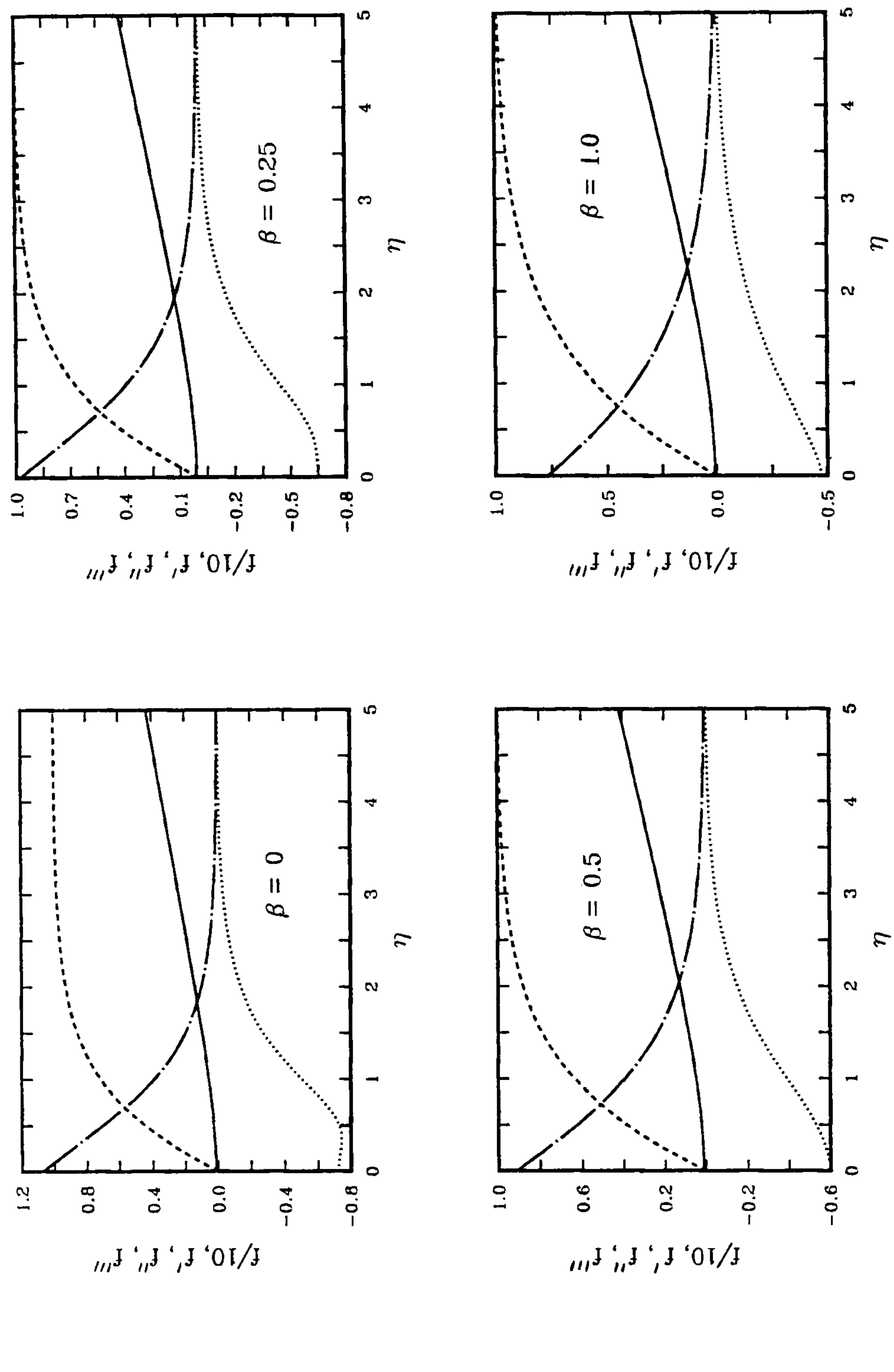

Figure 4. The function f and its derivatives for $x^{-2a}/K=1$, $b=-0{\cdot}1$ and $\beta=0, \frac{1}{4}, \frac{1}{2}$ and 1: ——, $f/10$; - - -, f'; —·—, f''; · · ·, f'''

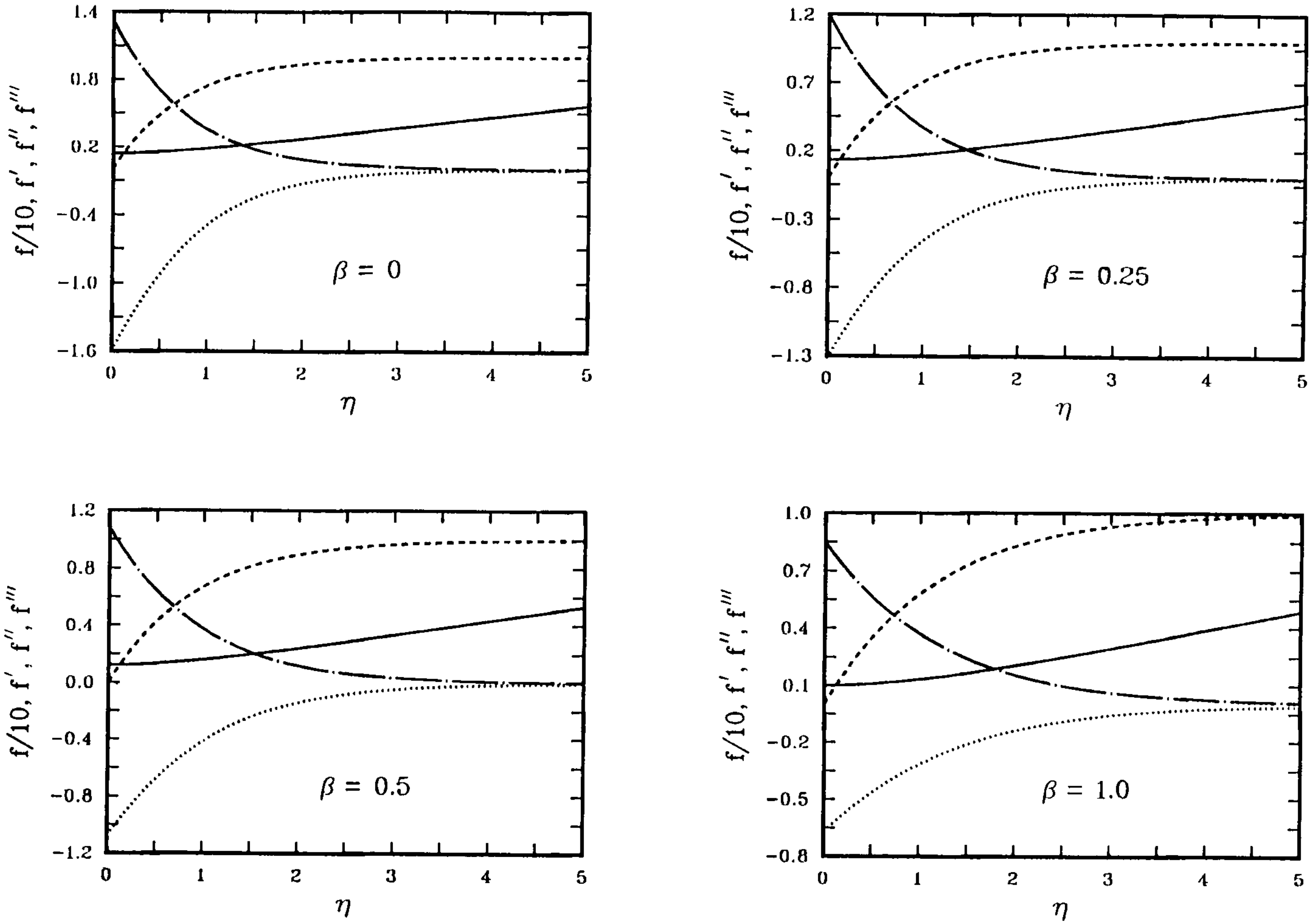

Figure 5. The function f and its derivatives for $x^{-2a}/K=1$, $b=-1$ and $\beta=0, \frac{1}{4}, \frac{1}{2}$ and 1: ——, $f/10$; - - -, f'; -·-, f''; . . . , f'''

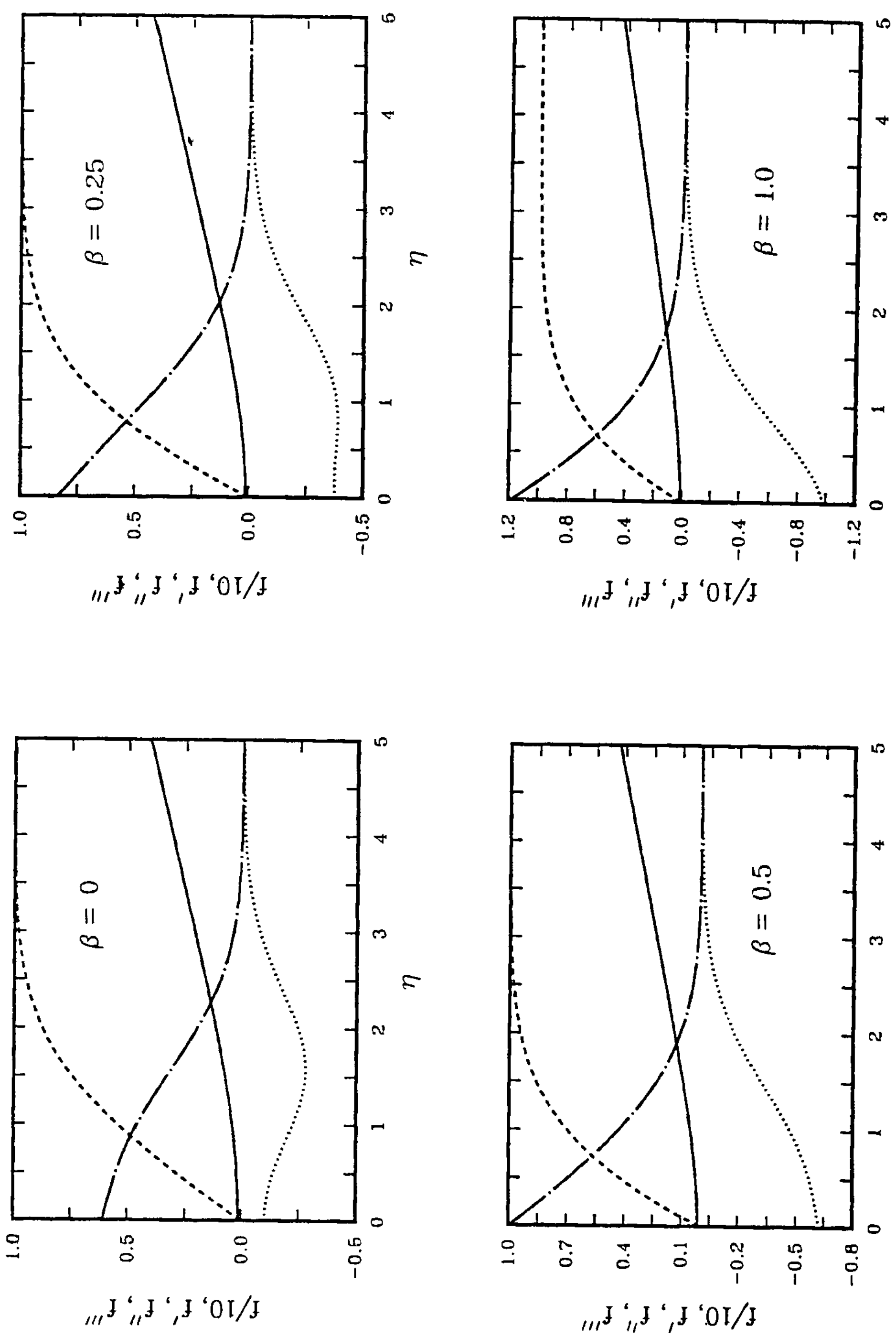

Figure 6. The function f and its derivatives for $x^{-2a}/K=10$, $b=-0{\cdot}1$ and $\beta=0, \frac{1}{4}, \frac{1}{2}$ and 1: ——, $f/10$; - - -, f'; – · –, f''; . . . , f'''

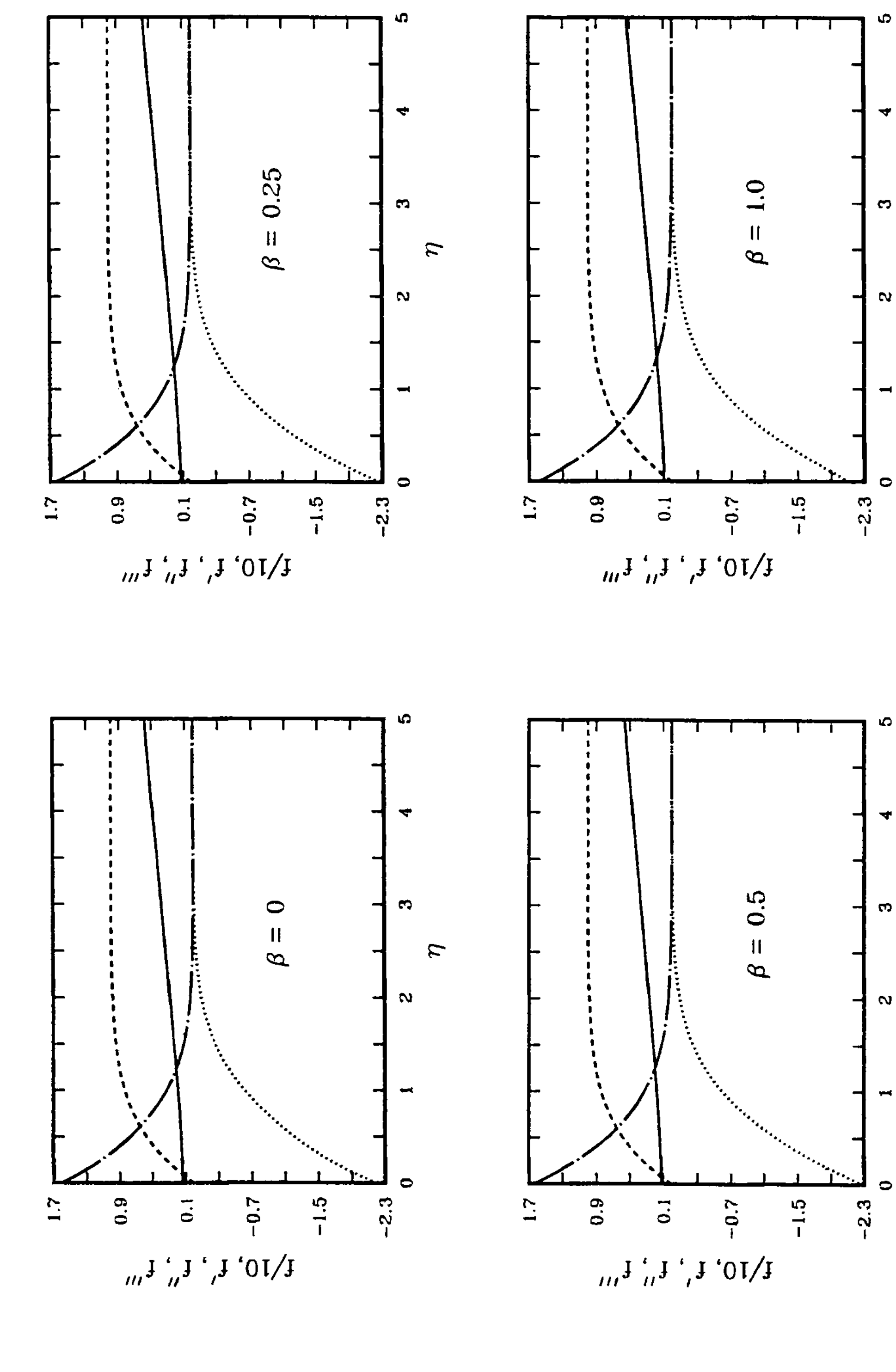

Figure 7. The function f and its derivatives for $x^{-2a}/K=10$, $b=-1$ and $\beta=0, \frac{1}{4}, \frac{1}{2}$ and 1: ——, $f/10$; - - -, f'; -·-, f''; · · ·, f'''

reduces from four to three and no orthonormalization is required for solution. In fact, a Newton–Raphson iteration is sufficient to solve the Falkner–Skan problem.

Figures 2–7 show the variation of f and its first three derivatives with respect to η for $\beta=0$ (flat plate at zero incidence), $\beta=\frac{1}{4}$ and $\frac{1}{2}$ (wedges of angles $\pi/4$ and $\pi/2$) and $\beta=1$ (two-dimensional stagnation point flow) for three values of x^{-2a}/K and two values of the suction parameter b. It is clear from these figures that the value of η_∞ used was adequate to simulate $\eta=\infty$. In fact, these figures show the results only up to $\eta=5$ or 10, while the value of η_∞ used was 20. It may be noted that while the behaviour of f and f' does not change much with the wedge angle, that of f'' and f''' changes considerably at fixed values of x^{-2a}/K and b. Large values of $f''(0)$ and $-f'''(0)$ may be noted for $\beta=0$ in Figure 2. These values drop considerably as suction increases (see Figure 3). As x^{-2a}/K increases and when the suction parameter b is small, f''' passes through a minimum for $\beta\leqslant\frac{1}{4}$ (see Figures 4 and 6), but for $\beta\geqslant\frac{1}{2}$ or for large suction it increases monotonically to zero as η increases. At a much smaller value of x^{-2a}/K, f'' and f''' for $\beta\leqslant\frac{1}{4}$ also vary monotonically with η, as shown in Figure 2 for $x^{-2a}/K=0{\cdot}1$. From these figures we conclude that the boundary layer thickness increases with β at small values of x^{-2a}/K but decreases with β at large values of x^{-2a}/K. Moreover, $f''(0)$ increases with β at large values of x^{-2a}/K but decreases with β at small values of x^{-2a}/K. Figure 7 shows that at large values of x^{-2a}/K ($=10$ here) and with the suction parameter $b=-1$, the wedge angle β has only a small effect on the characteristics of the flow.

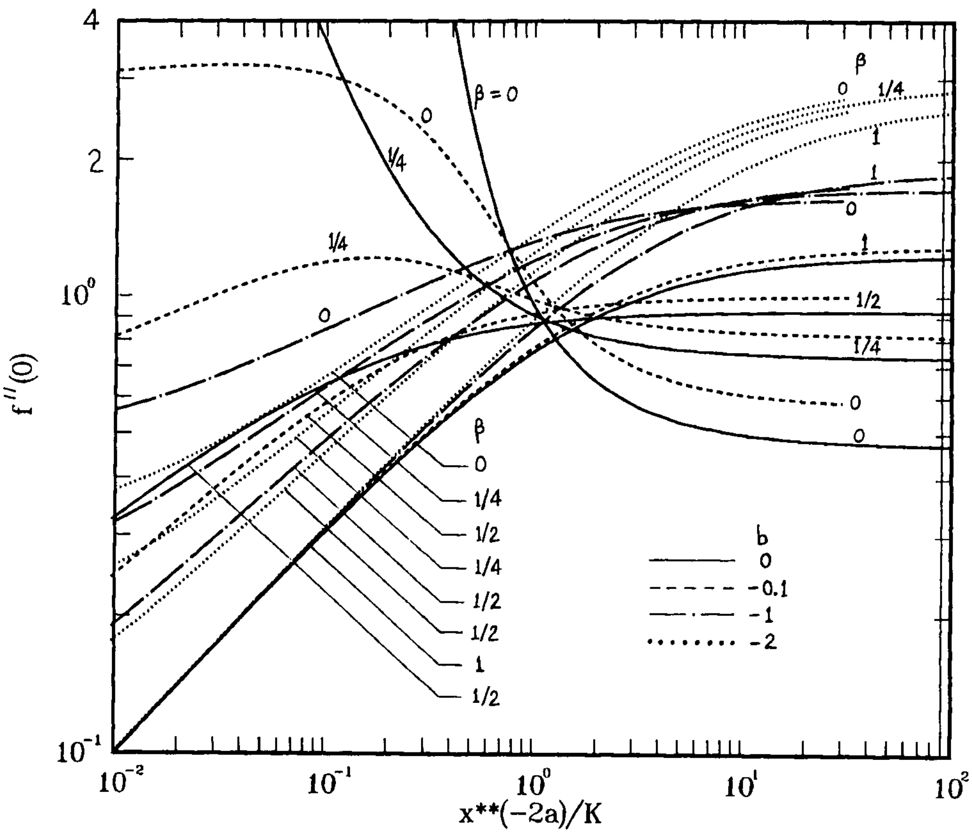

Figure 8. Wall shear stress as a function of x^{-2a}/K for various values of β and suction parameter b

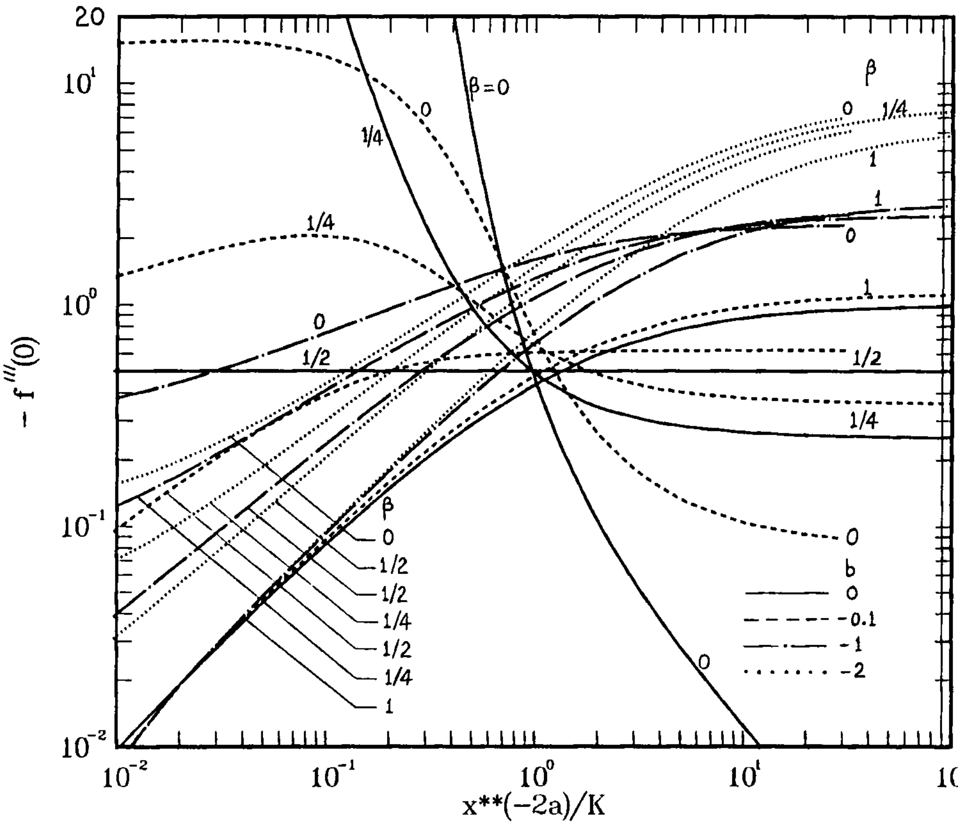

Figure 9. Derivative of wall shear stress as a function of x^{-2a}/K for various values of β and suction parameter b

It is found that with no suction ($b=0$), values of $f''(0)$ corresponding to $\beta \leqslant \frac{1}{4}$ increase rapidly as Kx^{2a} increases beyond about one. This is evident from Figure 8, which shows the variation of $f''(0)$ with x^{-2a}/K for $b=0$. It may be pointed out that $f''(0)$ determines the shear stress at the wall and is therefore physically important. It is clear from Figure 8 that for small wedge angles ($\beta \leqslant \frac{1}{4}$) even a small amount of suction ($b=-0{\cdot}1$) reduces the skin friction considerably for $x^{-2a}/K<1$ and increases it for $x^{-2a}/K>1$, while for larger wedge angles ($\beta \geqslant \frac{1}{2}$) the effect of suction is much smaller. For the stagnation point flow ($\beta=1$) the skin friction is not affected by suction for $x^{-2a}/K<0{\cdot}1$ but increases with suction for $x^{-2a}/K>0{\cdot}1$. A similar effect of suction is found for the derivative of the skin friction, $f'''(0)$, as shown in Figure 9 for various values of the suction parameter b and wedge angle β.

A comparison between the present $f''(0)$-values and those of Massoudi and Ramezan[1] obtained from a perturbation analysis shows that our results match exactly their results for $Kx^{2a} \leqslant 0{\cdot}01$. However, the present results differ significantly as Kx^{2a} increases or as x^{-2a}/K decreases, and it is this region of flow that is of interest. The perturbation analysis of Massoudi and Ramezan[1] is therefore of academic interest only.

Negative values of the parameter b yield a unique solution for a given β and Kx^{2a} as discussed above. However, positive values of b, pertaining to injection, lead to multiple solutions. A detailed analysis is currently under way in terms of the stability of the various solutions and related issues. For the flow of a Newtonian fluid over a wedge it is known that injection yields no stable solution.

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